Final Examination

Time:	10:00–13:00, December 29, 2021.	Course name:	*Algebra I*
Degree:	MMath.	Year:	1st Year, 1st Semester; 2021–2022.
Course instructor:	Ramdin Mawia.	Total Marks:	50.

Attempt any three of the following problems, including problem n° 2. All rings are commutative with identity, and all ring morphisms take identity to identity.

Indian Statistical Institute Bangalore Centre

Rings and modules

1. Define and construct the tensor product of modules. State its universal property. **3+7+4+6 =20**

   (a) Define restriction and extension of scalars for modules. Let $A \to B$ be a ring morphism and let M be an A-module and N be a B-module. Show that there is a natural isomorphism of abelian groups

   $$\operatorname{Hom}_B(B \otimes_A M, N) \cong \operatorname{Hom}_A(M, N).$$

   Is it an isomorphism of A-modules? Justify.

   (b) Let A be a ring and $A[X] \to A$ be the evaluation map at 0 (i.e., $f(X) \mapsto f(0)$), so that A is an $A[X]$-algebra. Is it true that $A \otimes_{A[X]} A \cong A$? Justify your claim.

   (c) Let A be an integral domain with quotient field K and let B be a K-algebra. Let $M = K \otimes_A B$, so M is an A-algebra, and by extension of scalars, a K-algebra as well. Is it always true that

   i. $M \cong B$ considering both M and B as A-algebras?

   ii. $M \cong B$ considering both M and B as K-algebras?

   Give justifications.

2. Define Noetherian rings and modules. **2+3+5 =10**

   (a) Is it true that every subring of a Noetherian ring is Noetherian? Justify.

   (b) Let A be a ring, M be a Noetherian A-module and $I = \operatorname{Ann} M$ be the annihilator of M. Show that A/I is a Noetherian ring.

3. Let A be a ring. Define a short exact sequence of A-modules. When do we say that a short exact sequence is split? Let **3+8+9 =20**

   $$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

   be a short exact sequence of A-modules.

   (a) Let S be a multiplicative submonoid of A^*. Show that the sequence of A-modules

   $$0 \longrightarrow S^{-1}A \otimes_A L \xrightarrow{1\otimes f} S^{-1}A \otimes_A M \xrightarrow{1\otimes g} S^{-1}A \otimes_A N \longrightarrow 0$$

   is a short exact sequence. Here $1 \otimes f$ and $1 \otimes g$ are the A-linear morphisms induced by $(a/s, x) \mapsto (a/s) \otimes f(x)$ and $(a/s, x) \mapsto (a/s) \otimes g(x)$ respectively.

   (b) Suppose L, M and N are free of finite rank. Prove that the induced sequence

   $$0 \to N^\vee \xrightarrow{g^*} M^\vee \xrightarrow{f^*} L^\vee \to 0$$

   is a split short exact sequence. Here $M^\vee = \operatorname{Hom}_A(M, A)$ etc.

4. Decide whether the following statements are true or false, with brief justifications (counterexamples, proofs, or such and such a theorem implies this etc) (**any ten**): **20**

   (a) The polynomial ring $\mathbb{Z}[X]$ is isomorphic to the power series ring $\mathbb{Z}[[X]]$.

   (b) Let A be a UFD. A power series $a_0 + a_1X + \cdots \in A[[X]]$ is irreducible in $A[[X]]$ if and only if a_0 is irreducible in A.

   (c) The power series ring $\mathbb{Q}[[X]]$ is a PID.

   (d) The power series ring $\mathbb{Z}/25\mathbb{Z}[[X]]$ is a complete local ring.

(e) Let $a_n = 5n$ if $5 \nmid n$ and $a_n = 2$ if $5|n$. Then the Weierstrass degree of the power series $\sum_{n=1}^{\infty} a_n X^{n-1} \in \mathbb{Z}_{\langle 5 \rangle}[[X]]$ is 4 and its Weierstrass polynomial is $5 + 5X + 5X^2 + 5X^3 + X^4$.

(f) If A is a subring of $\mathbb{Z}[X]$ which strictly contains $\mathbb{Z}$ (i.e., $\mathbb{Z} \subsetneq A \subset \mathbb{Z}[X]$), then $\mathbb{Z}[X]$ is a finitely generated A-module.

(g) For any ring morphism $A \to B$, we have $A[X] \otimes_A B \cong B[X]$ as A-modules.

(h) If A is a local ring, then $A[X]/\langle X^n \rangle$ is a local ring for each positive integer n.

(i) For any positive integers m and n, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ with $d = \gcd(m, n)$.

(j) There is a $\mathbb{Z}$-module M such that the sequence $0 \to \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow M \to 0$ is split short exact.

(k) In a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$, if M' and M'' are finitely generated then so is M.

(l) If S is a multiplicative subset of an integral domain A with $0 \notin S$, then $S^{-1}A$ is a local ring.

(m) If I is an ideal of a Noetherian ring A, then A/I is a Noetherian ring.

(n) The polynomial $X^3 + 2X + 1$ is irreducible in $\mathbb{Z}[X]$.